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SPECTRAL ANALYSIS AND THE APPLICATION OF FILTERS IN THE EXAMINATION OF BUSINESS CYCLES

INTRODUCTION

The spectral analysis enables us to divide a given time series into components characterised by a different frequency of fluctuations. Therefore, it is possible to use it for the extraction of the cyclical component from macroeconomic data. The most popular method of analysing components with a specific frequency is the use of the Hodrick-Prescott, Baxter-King and Christiano-Fitzgerald filters. Each of these filters differs in structure and has its own advantages and disadvantages.

The subject of this paper is the presentation of the general idea of the spectral analysis and the details of the construction of Hodrick-Prescott, Baxter-King and Christiano-Fitzgerald filters. Then all the three filters will be applied to real GDP time series of Poland and Greece. First, to compare their effects, and then to determine the conditions under which the filter works the most effectively.

The first part of the paper presents the spectral analysis and the construction details of the Hodrick-Prescott, Baxter-King and Christiano-Fitzgerald filters. The second part presents the results of empirical studies on the example of Poland and Greece, on the basis of which the difference in the results obtained with the use of different filters is shown.

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1. Spectral analysis

This section will first present the general idea of the spectral analysis, followed by three most popular filters operating in the frequency domain: of Hodrick-Prescott (HP), Baxter-King (BK) and Christiano-Fitzgerald (CF). The first point was primarily prepared on the basis of works of Hodrick and Prescott[1], King and Rebelo[2], Baxter and King[3] and Christiano and Fitzgerald[4].

1.1. The idea of Spectral analysis

In the spectral analysis it is assumed that a given time series consists of components that periodically oscillate at different frequencies[5]. In this case, one can think of a time series as consisting of a sum of periodic functions. For this purpose, the sine and cosine functions shown in panel a) of Figure 1 are used. These functions are periodic, which means that $\sin(t) = \sin(t + 2\pi h)$, for $h = 1, 2, \ldots$, and therefore, they regularly repeat their form. By giving these functions appropriate coefficients it is possible to change the amplitude of their fluctuations as shown in panel b). In addition, adding a factor to the function argument changes the periodicity of the function – the greater its value, the more frequently the periodic repetitions occur. The idea of the period change is shown in panel c). It is also worth adding that the sine and cosine functions are characterised by the same period and amplitude, but they differ from each other by the phase shift, which makes them pass through the same cycle elements at different moments. Finally, summing the sine and cosine functions with different coefficients and periodicities, it is possible to model cyclical fluctuations with very different periodicity, variability and amplitude, as shown in panel d). When combining the sine

1 Hodrick, R.R., Prescott, E. 1997, Postwar U.S. Business Cycles: An Empirical Investigation. *Journal of Money, Credit, and Banking*, vol. 29, no. 1, pp. 1–16.

2 King, R., Rebelo, S. 1993. Low Frequency Filtering and Real business Cycles. *Journal of Economic Dynamics and Control*, no. 17, North-Holland, pp. 207–231.

3 Baxter, M., King, R. 1999. Measuring Business cycles: Approximate Band-Pass Filters for Economic Time Series. *The Review of Economics and Statistics*, vol. 81, no. 4, pp. 575–593.

4 Christiano, L., Fitzgerald, T. 1998. The Business Cycle: It's Still a Puzzle. *Federal Reserve Bank of Chicago Economic Perspectives*, vol. 22, no. 4, pp. 56–83; Christiano, L., Fitzgerald, T. 2003. The Band Pass filter. *International Economic Review*, vol. 44, no. 2, 435–465.

5 More on the topic of spectral analysis can be found in Hamilton, J. 1994. *Time Series Analysis*. Princeton, New Jersey: Princeton University Press.

and cosine functions with different periodicity their weights in the resulting function depend on the absolute value of their parameters.

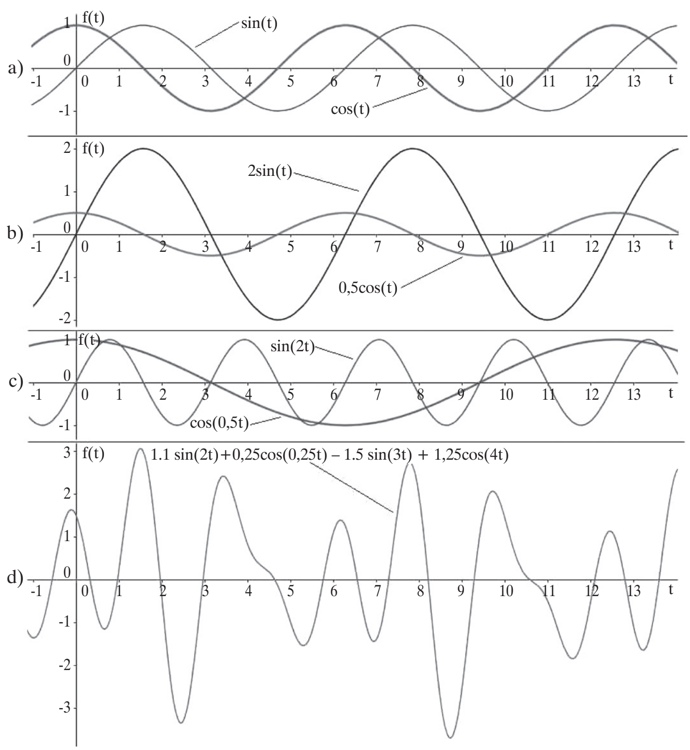
When we consider the functions in the following form:

$$\sin(t\omega) \text{ oraz } \cos(t\omega),\ t = 1, 2, \dots \tag{1}$$

the oscillation period is expressed in units of time, and frequency ω determines after how many units of time there will be a full period – after what amount of time the increase by 2π will occur. Assuming there are two points in time t_1 and t_2 in the same phase of the cycle, we can write: $t_2\omega - t_1\omega = 2\pi$, and after dividing both sides of the equation by frequency ω we get: $t_2 - t_1 = 2\pi/\omega$. Thus, in the case of the function sin $(t\omega)$ and cos $(t\omega)$ the period of oscillation in time units is $2\pi/\omega$, and therefore the parameter ω is referred to as the frequency of oscillation.

Figure 1

Trigonometric functions



Source: own arrangement.

With the above information it is possible to present the basic idea of the spectral analysis. Assuming that there is a time series y_t, $t = 1, 2, T$, where T is an even number, it is possible to decompose it into $T/2$ of possible periodic functions, each of which is characterised by a different frequency of fluctuations ω. Let $\omega_j = 2\pi j/T$, for $j = 1, 2, \ldots, T/2$ and a_j and b_j for $j = 1, 2, \ldots, T/2$ denote parameters of the cosine and sine functions. Then it is possible to write the time series y_t in the form[6]:

$$y_t = a_1\cos(t\omega_1) + b_1\sin(t\omega_1) + \ldots + a_{T/2}\cos(t\omega_{T/2}) + b_{T/2}\sin(t\omega_{T/2}) \tag{2}$$

It is possible to find all the parameters of a_j and b_j for $j = 1, 2, \ldots, T/2$ of this equation using OLS. When the Tx$T/2$ matrix of explanatory variables X, the vector T x 1 of the explanatory variables Y and the vector T x 1 of the regression coefficients are given by:

$$X = \begin{bmatrix} \cos(\omega_1) & \sin(\omega_1) & \cdots & \cos(\omega_{T/2}) & \sin(\omega_{T/2}) \\ \cos(2\omega_1) & \sin(2\omega_1) & \cdots & \cos(2\omega_{T/2}) & \sin(2\omega_{T/2}) \\ \vdots & \vdots & & \vdots & \vdots \\ \cos(T\omega_1) & \sin(T\omega_1) & \cdots & \cos(T\omega_{T/2}) & \sin(T\omega_{T/2}) \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}, \beta = \begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ a_{T/2} \\ b_{T/2} \end{bmatrix}, \tag{3}$$

the regression equation is given by:

$$Y = X\beta + \varepsilon. \tag{4}$$

Due to the fact that the number of explanatory variables is T, the residual is 0, and then β can be calculated as $(X'X)^{-1}X'Y = X^{-1}Y$. Obviously, the calculation of the parameters is only possible in theory. It results from the fact that since the amount of all potential frequencies is infinite we would also need an infinite number of observations for the exact calculation, which is by no means possible. Because of this problem in the spectral analysis, equation (2) is transformed into the following integral:

$$y_t \int_0^{\pi} [a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)]d\omega, \tag{5}$$

where $a(\omega)$ and $b(\omega)$ are functions of ω. Every weakly stationary process can be expressed by an equation[7] (5). This equation allows us to show the

[6] Christiano, L., Fitzgerald, T. 1998, *op. cit.*

[7] Koopmans, L. 1974. *The Spectral Analysis of Time Series*. New York: Academic Press.

time series y_t as the sum of the components oscillating between $2\pi/\omega$ for ω belonging to the range $[0, \pi]$.

1.2. Hodrick-Prescott filter

The filter constructed by Hodrick and Prescott[8] (HP) methodologically corresponds to the Lucas's definition of the business cycle, which defines cyclical fluctuations as deviations from the trend[9]. Initially, the authors developed this filter in the time domain, therefore, in the considerations in this section the perspective of time is adopted. Because the HP filter is a high-pass filter, in the frequency domain is also described when discussing the Baxter-King band pass filter. The starting point is the assumption that the real GDP time series consists of two components:

$$Y_t = Y_{Ct} + Y_{Nt} \text{ dla } t = 1, 2, \ldots, T, \tag{6}$$

where Y_{Ct} is the cyclical component of GDP in period t, and Y_{Nt} – the trend value in period t. This means that the filter ignores the seasonal component that has to be removed in a separate procedure.

The application of the Lucas definitions for the HP filter takes place by finding the solution to the following mathematical programming problem:

$$\min_{Y_{Nt}} \left\{ \sum_{t=1}^{T} (Y_t - Y_{Nt})^2 + \lambda \sum_{t=1}^{T} [(Y_{Nt+1} - Y_{Nt}) - (Y_{Nt} - Y_{Nt-1})]^2 \right\}. \tag{7}$$

Hodrick-Prescott filter takes the sum of the squares of the second differences of the time series as the criterion of smoothness. λ is a positive parameter penalising variance in the second differences. When $\lambda \to \infty$, the solution to the problem approaches the OLS estimation. $Y_{Ct} = Y_t - Y_{Nt}$ is a cyclical component and it is assumed that its expected value tends to zero with the prolongation of the time sample. Due to its design, the HP filter can be considered a generalisation of the exponential smoothing procedures formulated by Brown[10]. It is worth noting that trend elimination techniques

8 Hodrick, R., Prescott, E., *op. cit.*

9 Lucas, R. 1977. Understanding business cycle. In: Brunner, K., Meltzer, A. eds. *Stabilization of the domestic and international economy*. Amsterdam, North-Holland: Carnegie-Rochester conference Series on Public Policy, vol. 5, pp. 7–29.

10 Brown, R. 1962. *Smoothing, Forecasting and Prediction of Discrete Time Series*. New Jersey: Prentice-Hall.

very similar to the Hodrick-Prescott filter were previously used in actuarial and ballistic sciences[11].

Assuming that the cyclical components and the second differences of the series are independent with identical distributions with zero mean and variances amounting respectively to σ_c^2 and $\sigma_{\Delta^2 Y_t}^2$, the parameter penalising variance is given by: $\lambda = \sigma_C^2 / \sigma_{\Delta^2 Y_t}^2 \leftrightarrow \sqrt{\lambda} = \sigma_C / \sigma_{\Delta^2 Y_t}$. The variance penalising parameter is the only part of the filter that needs to be determined by the researcher. Hodrick and Prescott assumed that the moderate cyclical component had a variation of about 5% while the second difference of only (1/8)%. On this basis, they propose $\sqrt{\lambda} = 5/(1/8) = 40 \rightarrow \lambda = 1600$ as the optimal value of the penalising parameter for the quarterly data.

The value of the parameter $\lambda = 1600$ was developed for quarterly data and is a value that has gained widespread acceptance in both economic and econometric literature. However, the problem arises when it is necessary to specify the parameter for annual data. Then Hodrick and Prescott propose $\lambda = 100$, which was accepted in the study of Backus and Kehoe[12]. Rawn and Uhlig[13] argue that for a better comparison with the results for quarterly data, the value of λ should amount to 6.25, while Cooley and Ohanian[14] and Correia, Neves, and Rebelo[15] suggest that this figure is equal to 400. In turn, Baxter and King[16] opt for a value of 10 for λ.

The cyclical component obtained with the HP filter can be also represented as a moving average[17]:

$$HP(L) = \frac{\lambda(1-L)^2(1-L^{-1})^2}{1+\lambda(1-L)^2(1-L^{-1})^2}, \tag{8}$$

where L is the polynomial lag operator. In addition, when $T \rightarrow \infty$ the solution (7) can be found in the frequency domain. King and Rebelo showed that in

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11 Stigler, S. 1978. Mathematical statistics in the Early States. *Annals of Statistics*, vol. 6, pp. 239–265.

12 Backus, D., Kehoe, P. 1992. International Evidence of the Historical Properties of Business Cycles. *American Economic Review*, vol. 82, no. 4, pp. 864–888.

13 Ravn, M., Uhlig, H. 2002. On adjusting the Hodrick-Prescott filter for the frequency of observations. *Review of Economics and Statistics*, vol. 84, no. 2, pp. 371–380.

14 Cooley, T., Ohanian, L. 1991. The Cyclical Behavior of Prices. *Journal of Monetary Economics*, vol. 28, no. 1, pp. 25–60.

15 Correia, I., Neves, J., Rebelo, S. 1992. Business cycle from 1850-1950: New Facts about Old Data. *European Economic Review*, vol. 36, pp. 459–467.

16 Baxter, M., King. R., *op. cit.*

17 King, R., Rebelo, S., *op. cit.*

such a situation the expression for the frequency response function ($\psi(\omega)$) can be expressed as:

$$\psi(\omega) = \frac{4\lambda[1-\cos(\omega)]^2}{1-4\lambda[1-\cos(\omega)]^2}. \tag{9}$$

The weight (γ_j) in the moving average for the HP filter can be determined by using the inverse Fourier transformation:

$$\gamma_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{4\lambda[1-\cos(\omega)]^2}{1-4\lambda[1-\cos(\omega)]^2} e^{-i\omega j} d\omega. \tag{10}$$

The HP filter is a high-pass filter that passes only a component that is characterised by the frequency of fluctuation above limit value χ. The relationship between the cut-off frequency and the value of parameter λ can be determined using equation (9) and is given by[18]:

$$\chi = \frac{\pi}{\arcsin\left(\frac{1}{2}\lambda^{-\frac{1}{4}}\right)} \leftrightarrow \lambda = \left[2\sin\left(\frac{\pi}{\chi}\right)\right]^{-4}. \tag{11}$$

The HP filter is a filter that passes this part of the fluctuation band that is above cut-off frequency χ. The details of the operation of high-pass filters are presented in section 1.3. However, at this stage one may notice that the HP filter treats the seasonal and cyclical components equally. This means that this filter, in order to obtain a cyclical component, should be applied only to data from which the seasonal component has been removed.

1.3. Baxter-King filter

The second most popular method of extracting the cyclical component and the trend from macroeconomic data is the Baxter and King[19] (BK) filter. This filter (and the Christiano-Fitzgerald filter) is a methodological equivalent of Burns and Mitchell's business cycle definition, according to which these are fluctuations within a certain frequency band – over one year and less than ten or twelve years[20]. The idea of the filter is based on a moving average. If the moving average is used for a series of real GDP, Y_t, then a new time series will be produced:

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18 Kowal, P. 2005. *Optimal filtering*. Available at: https://ideas.repec.org/s/wpa/wuwppr.html.

19 Baxter, M., King, R., *op. cit.*

20 Burns, A., Mitchell, W. 1946. *Measuring business cycles*. New York: NBER.

$$Y_t^* = \sum_{k=-K}^{K} a_k Y_{t-k}, \tag{12}$$

where k is the number of lags, a_k is the weight for the value of real GDP delayed by k periods. The moving average can also be expressed using a polynomial lag operator L:

$$a(L) = \sum_{k=-K}^{K} a_k L^k, \tag{13}$$

and then: $L^K y_k = y_{t-k}$. If the average is symmetric then: $a_k = a_{-k}$ for $k = 1, 2, \ldots, K$. Baxter and King prove that when the moving average weights add up to zero:

$$\sum_{h=-K}^{K} a_h = 0, \tag{14}$$

then the moving average guarantees the stationarity of the series irrespective of whether the output series is characterised by a deterministic or stochastic trend.

If a series is stationary with an average of 0, then it can be expressed by the Cramer representation as follows:

$$Y_t = \int_{-\pi}^{\pi} \xi(\omega)\, d\omega. \tag{15}$$

Thus, the representation of a series is an integral of random periodic components, where:

$$EV\{\xi(\omega_1)[\xi(\omega_2)]\} = 0 \ \ dla \ \ \omega_1 \neq \omega_2, \tag{16}$$

and ω is a specific frequency. The filtered series can be written down as:

$$Y_t^* = \int_{-\pi}^{\pi} \alpha(\omega)\xi(\omega)\, d\omega, \tag{17}$$

where:

$$\alpha(\omega) = \sum_{h=-K}^{K} a_h e^{-i\omega h} \tag{18}$$

is a frequency-response function of the linear filter, and on the basis of (16) it can be shown that the variance of the filtered series is given by the following formula:

$$VAR(Y_t^*) = \int_{-\pi}^{\pi} [\alpha(\omega)]^2 f_y(\omega) d\omega. \tag{19}$$

$[\alpha(\omega)]^2$ is a transfer (squared gain) function of the linear filter for frequency ω, while $f_y(\omega) = VAR[\xi(\omega)]$ is the spectral density for series y of frequency ω. The response frequency function is zero for $\omega = 0$ (for frequency 0) only when the sum of the filter weights is zero:

$$\alpha(\omega) = \sum_{h=-K}^{K} a_h e^{-i\omega h} = 0 \leftrightarrow \sum_{k=-K}^{K} a_k = 0. \tag{20}$$

The BK filter is a band pass filter. Before describing its operation, it is worth analysing the case of a simpler low pass (LP) filter, this filter passes only a 'slowly moving' part of the data – below a specified frequency.

The ideal symmetrical LP filter, which passes only frequencies from the range $\underset{\smile}{\omega} \leq \omega \leq \underset{\smile}{\omega}$, has a frequency response function $\beta(\omega) = 1$ for $|\omega| \leq \underset{\smile}{\omega}$ and $\beta(\omega) = 0$ for $|\omega| > \underset{\smile}{\omega}$ where $\underset{\smile}{\omega}$ is the cut-off frequency. Symmetrical weights also imply that $\beta(\omega) = \beta(-\omega)$. The representation of the ideal low pass filter in the time domain is:

$$b(L) = \sum_{h=-\infty}^{\infty} b_h L^h, \tag{21}$$

where b_h denotes weights. These weights can be calculated using the inverse Fourier transformation for the following frequency function:

$$b_h = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(\omega) e^{-i\omega h} d\omega. \tag{22}$$

Baxter and King[21] proved that:

$$b_0 = \frac{\underset{\smile}{\omega}}{\pi} \wedge b_h = \frac{\sin\left(h\underset{\smile}{\omega}\right)}{h\pi} \quad dla \;\; h = 1, 2, \ldots \tag{23}$$

This ideal low pass filter can only work when $h = \infty$. Therefore, Baxter and King propose approximation using a finite moving average $a(L) = \sum_{h=-K}^{K} a_h L^H$ for which the frequency-response function is given by $\alpha_K(\omega) = \sum_{h=-K}^{K} a_h e^{-i\omega h}$.

In order to choose the optimum approximation of the ideal filter, we need to choose the weights a_h to minimize the following expression:

21 Baxter, M., King, R., *op. cit.*

$$Q = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\delta(\omega)]^2 d\omega, \tag{24}$$

where $\delta(\omega) = \beta(\omega) - \alpha_K(\omega)$ is a discrepancy resulting from the approximation for frequency ω. The optimal approximation of the ideal filter for K lags is achieved by by simply truncating the ideal filter's weights b_h at lag K. Therefore, the optimum low pass filter gives $a_h = b_h$ for $h = 0, 1, 2, \ldots, K$ and for $a_h = 0$. Weights b_h are given by (22).

In an analogous way, the ideal high pass filter (HPass) and its approximation can be constructed. This filter would only pass a 'fast-moving' part of the time series – above a specified frequency. The ideal HPass filter would be characterised by frequency-response function γ_h. We can obtain weights c_h for this filter in a similar way. In this way, a high-pass filter can be described in the frequency domain. Knowing the form of ideal and approximate high and low pass filters, and the resulting weights, it is possible to construct the Baxter-King (BK) band pass filter.

The ideal band pass filter only passes frequencies from the specified range: $\underbrace{\omega} \leq |\omega| \leq \overbrace{\omega}$, where $\underbrace{\omega}$ is the cut-off frequency of the LP filter, while $\overbrace{\omega}$ is the cu-toff frequency of the HPass filter. The frequency-response function for the BK filter is given by $\gamma(\omega) - \beta(\omega)$ and assumes a value of 1 for $\underbrace{\omega} \leq |\omega| \leq \overbrace{\omega}$ and 0 for all other frequencies. Weights of approximation to the ideal band pass filter – when tuned due to the number of lags and leads of the Baxter-King filter – are given by $c_h - b_h$. So the BK can be expressed as:

$$BK(L) = \sum_{h=-K}^{K} (c_h - b_h)L^h. \tag{25}$$

The BK filter requires the introduction of K lags and leads, which means that $2K$ observations are lost. Yet, with the increase in the number of lags and leads, the approximation (of the ideal filter) obtained with the BK filter is better. However, this is done at the expense of losing observations. Baxter and King experimented with different levels of K tuning and concluded that 12 is optimum for quarterly data, while 3 is for annual data. This is the same number of lags and leads that should be removed from the Hodrick-Prescott filter to avoid untypical results observed by Baxter and King at the beginning and end of the filtered series.

It is worth showing the relationship between periodicity and frequency. The relationship is described by p or $f = 2\pi/\omega$, where p or f denotes one period – a quarter/year. p and f define cut-off frequencies (and corresponding periods) between those passed and retained by the filter. Therefore, the use of the BK filter requires the specification of the cut-off frequencies. The filter will stop a component belonging to a given frequency band as a cyclical component, or else – a component with a certain periodicity.

There is another element of the BK filter that deserves to be emphasised. As mentioned earlier, BK filter is characterised by weights that add up to zero, which guarantees stationarity of the obtained cyclical component. In addition, BK is a symmetric filter ($\beta(\omega) = \beta(-\omega)$), which in turn guarantees the removal of both linear and quadratic trend from the time series. The use of symmetrical weights has yet another advantage, namely the lack of the phase shift. For these reasons, the BK filter can be easily applied to a wide range of macroeconomic time series.

1.4. Christiano-Fitzgerald filter

Christiano and Fitzgerald[22], just like Baxter and King, tried to find the optimal approximation of the ideal filter. The authors assumed that a given stochastic process y_t could be divided into two parts:

$$y_t - y_t^* + \tilde{y}_t, \tag{26}$$

where process y_t^* contains only this part of frequencies that fall within the range $\left\{\left(\underbrace{\omega}, \overbrace{\omega}\right) \cup \left(-\underbrace{\omega}, \overbrace{-\omega}\right)\right\} \in (-\pi, \pi)$, while $\tilde{y}_t$ only the components out of that range, where $0 < \underbrace{\omega} \leq \overbrace{\omega} \leq \pi$.

For example, following Christiano and Fitzgerald[23], the cyclical GDP component is in the fluctuation band between 1.5 and 8 years. In the terminology of fluctuation frequencies for quarterly data, it can be written that it is a fraction ω within the range between $\underbrace{\omega} = 2\pi/32$ and $\overbrace{\omega} = 2\pi/6$. Referring to equation (5), we look for the cyclical component of series y_t given by:

$$y_t^* = \int_{\underbrace{\omega}}^{\overbrace{\omega}} [a(\omega)\cos(t\omega) + b(\omega)\sin(t\omega)]d\omega. \tag{27}$$

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22 Christiano, L., Fitzgerald, T. 2003, *op. cit.*

23 Christiano, L., Fitzgerald, T. 1998, *op. cit.*

Searching for a part of the series located in the above-described frequency band, the authors suggest the use of the ideal filter given by[24]:

$$y_t^* = B(L), \tag{28}$$

where $B(L) = \sum_{j=-\infty}^{\infty} B_j L^j$, $L^t y_t \equiv y_{t-1}$. For this specification $B(e^{-i\omega}) = 1$, for $\omega \in \left(\underbrace{\omega}, \overbrace{\omega}\right) \cup \left(-\underbrace{\omega}, \overbrace{-\omega}\right)$ and is equal to 0 for all others. In such a situation $\underbrace{\omega} > 0$ implies $B(1) = 0$. Once again there is a problem connected with the fact that the ideal filter requires an infinite number of observations and thus an approximation is needed.

Christiano and Fitzgerald focused their attention on finding an optimal approximation of a series that can be treated as realization a random walk process. This means that if a given macroeconomic series exhibits other properties, it is necessary to bring the series to exactly such a form (e.g. by removing a drift or a deterministic trend). Using $\widehat{y_t^*}$ as a symbol for the approximation of the ideal filter, we can write it as:

$$\widehat{y^*} = P[y^* \mid y], \tag{29}$$

and then for each observation we can write: $\widehat{y_t^*} = P[y_t^* \mid y]$ for $t = 1, \ldots, T$. Therefore, for each t the solution can be written as:

$$\widehat{y_t^*} = \sum_{j=-f}^{p} \hat{B}_j^{p,f} y_{t-j}, \tag{30}$$

where: $f = T - t$ and $p = t - 1$. In this case $\hat{B}_j^{p,f}$ solves the problem of minimizing mean square errors:

$$\min_{\hat{B}_j^{p,f},\, j=-f,\ldots,p} E\left[\left(y_t^* - \widehat{y_t^*}\right)^2 \mid y\right]. \tag{31}$$

The above problem can be presented in the frequency domain as follows:

$$\min_{\hat{B}_j^{p,f},\, j=-f,\ldots,p} \int_{-\pi}^{\pi} \left| B(e^{i\omega h}) - \hat{B}_j^{p,f}(e^{-i\omega h}) \right|^2 f_y(\omega)\, d\omega, \tag{32}$$

where $f_y(\omega)$ is the spectral density of y_t, while:

[24] Sargent, T. 1987. *Macroeconomic Theory*, 2nd ed. London: Academic Press.

$$\hat{B}_j^{p,f}(L) = \sum_{j=-f}^{p} \hat{B}_j^{p,f} L^j \wedge L^h y_t \equiv y_{t-h}. \tag{33}$$

When series y_t is a random walk, one can find $\widehat{y_t^*}$ using a moving average of the observations:

$$\begin{aligned} \widehat{y_t^*} = B_0 y_t + B_1 y_{t+1} + \cdots + B_{T-t-1} y_{T-1} + \hat{B}_{T-t-1} y_{t-1} + \\ + B_1 y_{t-1} + \cdots + B_{t-2} y_2 + \hat{B}_{t-1} y_1 \end{aligned} \tag{34}$$

for $t = 3, 4, \ldots, T-2$, where weights are given by:

$$B_0 = \frac{\overset{\frown}{\omega} - \underset{\smile}{\omega}}{\pi} \wedge B_J = \frac{\sin\left(j\overset{\frown}{\omega}\right) - \sin\left(j\underset{\smile}{\omega}\right)}{\pi j} \; dla \;\; j \geq 1. \tag{35}$$

and

$$\hat{B}_{T-t} = \frac{1}{2}B_0 - \sum_{j=1}^{T-t-1} B_J \wedge \hat{B}_{t-1} = \frac{1}{2}B_0 - \sum_{j=1}^{t-2} B_J. \tag{36}$$

If a given time series has the following representation:

$$y_t = y_{t-1} + \theta(L)\varepsilon_t \wedge E\varepsilon_t^2 = 1, \tag{37}$$

where $\theta(L)$ is the polynomial of degree q for lag operator L, then the spectral density function for y_t takes the form:

$$f_y(\omega) = \frac{g(\omega)}{(1 - e^{-i\omega})(1 - e^{i\omega})}, \tag{38}$$

where

$$g(\omega) = \theta(e^{-i\omega})\theta(e^{i\omega}) = c_0 + c_1(e^{-i\omega} + e^{i\omega}) + \cdots + c_q(e^{-i\omega q} + e^{i\omega q}). \tag{39}$$

The solution is given by[25]:

$$\hat{B}^{p,f} = A^{-1}d, \tag{40}$$

[25] The detailed calculation can be found in Christiano, L., Fitzgerald, T. 2003, *op. cit.*, pp. 443–446.

where

$$\hat{B}^{p,f} = \begin{bmatrix} \hat{B}^{p,f}_{p} \\ \hat{B}^{p,f}_{p-1} \\ \hat{B}^{p,f}_{p-2} \\ \vdots \\ \hat{B}^{p,f}_{-f} \end{bmatrix}, A = 2\pi \begin{bmatrix} F_{p-1}Q \\ \vdots \\ F_{-f+1}Q \\ F_{-f}Q \\ 1 \cdots 1 \end{bmatrix}, d = \begin{bmatrix} \int_{-\pi}^{\pi} \hat{B}(e^{-i\omega h}) g(\omega) d e^{i\omega(p-1)} d\omega \\ \vdots \\ \int_{-\pi}^{\pi} \hat{B}(e^{-i\omega h}) g(\omega) d e^{-\omega(-f+1)} d\omega \\ \int_{-\pi}^{\pi} \hat{B}(e^{-i\omega h}) g(\omega) d e^{i\omega(-f)} d\omega \\ 0 \end{bmatrix}, \quad (41)$$

and:

$$Q = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix}, \quad (42)$$

$$F = \left[\underbrace{0,\ldots,0}_{1\times(p-q-1-j)} \quad c \quad \underbrace{0,\ldots,0}_{1\times(p-q-1-f)} \right]$$

while $c = [c_q, c_{q-1}, \ldots, c_0, \ldots, c_{q-1}, c_q]$.

The above-mentioned solutions point to several significant aspects of the Christiano-Fitzgerald filter (CF). First, the CF filter is not symmetrical, which means that it uses the information contained in all observations to calculate values of the cyclical component for all periods. This allows the filter to produce cyclical values for all analysed periods. Unfortunately, due to using asymmetric weights, the filter does not have the trend removal properties. However, Christiano and Fitzgerald[26] show that the use of the filter for a series with other stochastic properties than those presented above leads to very small losses in the approximation of the ideal filter. A bigger problem is that the use of asymmetric weights can lead to phase shifts. It is possible to modify the filter to ensure symmetry of weights, but a result of such a procedure is the loss of data at the start and end of the analysed time series.

1.5. Filter comparison

The HP is a high-pass filter. This means that a researcher interested in a cyclical component of a time series should apply it only to seasonally

[26] Christiano, L., Fitzgerald, T. 2003, *op. cit.*

adjusted data. On the other hand, this condition is not required when the researcher is interested in a trend analysis. An advantage of this filter is that it does not lead to the loss of observations, but the observations at the beginning and the end of the filtered series behave in an unusual manner, which may lead to their rejection. This problem is particularly important when the researcher is interested in analysing the relationships that have taken place in the 'near past', which is essential for formulating recommendations for current economic policy. Last but not least, a great advantage of the HP filter is its significant popularity in applications that allow for broad comparability of results.

The BK and CF are band pass filters. The differences between these filters result from the very approach to the problem of minimizing the differences between the approximation and the ideal filter. In the case of the BK the mean square error $\left(\left|\hat{B}_j^{p,p}(e^{-i\omega h}) - B(e^{-i\omega h})\right|^2\right)$ is minimized, while CF minimizes the average square error weighted by the spectral density of the analysed series $\left(\left|B(e^{-i\omega h}) - \hat{B}_j^{p,f}(e^{-i\omega h})\right|^2 f_y(\omega)\right)$. In addition, Baxter and King made conditional optimization by imposing a constraint $\hat{B}_j^{p,p}(1) = 0$ that is absent in the work of Christiano and Fitzgerald. Due to the use of the constraint it is possible to remove the trend from the analysed series. Moreover, the CF filter uses all observations, so the filter is naturally asymmetrical. The symmetrical BK filter requires the abandonment of an equal number of observations at the start and end of the analysed time series. On the other hand, the symmetry of the BK guarantees no phase shift that is present in the case of the CF.

Due to these differences the BK and CF filters work better in different applications. Where the researcher wants to obtain the results that are not phase shifted and with the stationary cyclical component, it is recommended to use the BK filter. In addition, this filter can be applied to a wide range of series, which is an extraordinary advantage, especially when the stochastic properties of the analysed series cannot be clearly determined. In such a situation, the researcher must accept the loss of observation at the end and beginning of the analysed sample. This problem is absent in the case of CF filter, which also gives a better approximation of the ideal filter. Unfortunately, at the expense of a better approximation there is the possibility of phase shifts and the lack of properties removing the trend from the data. Therefore, this filter is best used when the properties of the time series are known and the proper transformation has been made on that basis.

2. Results of the application of the filters

This section presents the results of the application of spectral analysis to decompose GDP of European countries. The first subsection presents the used data and the properties of the time series. The second subsection compares the results obtained by means of using the three filters to seasonally adjusted and not adjusted data on the example of Greece and Poland.

2.1. Source of data and properties of time series

The study was conducted on the example of Poland and Greece. The study uses quarterly data on nominal GDP (in millions of euro) and price levels (2010 = 100) coming from Eurostat and covers the period from the first quarter of 2002 to the first quarter of 2016. Subsequently, the nominal GDP data were divided by the price level data to obtain the series of real GDP. The real GDP series is the subject of all presented analyses. The study used series containing the seasonal component as well as those from which the seasonal component was removed using X-13 ARIMA

The use of the CF filter requires knowledge of the stochastic properties of the analysed time series. Therefore, prior to filtration all series were subjected to the following unit root tests: ADF[27] (Augmented Dickey-Fuller test[28]) and KPSS[29] (Kwiatkowski-Phillips-Shmidt-Shin test). In both cases, the version of the test with the intercept (α) and the intercept and the linear trend ($\alpha + \beta t$) was used to determine whether the trend in the data is stochastic (S) or deterministic (D). When the inclusion of a deterministic trend causes the elimination of the unit root, following Nelson and Plosser[30] the trend is classified as deterministic. Otherwise, the hypothesis of the stochastic trend and the occurrence of the drift was accepted. The results of applying the test

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[27] More on this test can be found in Stadnytska, T. 2010. Deterministic or Stochastic Trend. Decision on the Basis of the Augmented Dickey-Fuller Test. *Methodology*, vol. 6, no. 2, pp. 83–92.

[28] Said, E., Dickey, D. 1984. Testing for Unit Roots in Autoregressive Moving Average Models of Unknown Order. *Biometrika*, vol. 71, no. 3, pp. 599–607.

[29] Kwiatkowski, D., Phillips, P., Schmidt, P., Shin, Y. 1992. Testing the Null Hypothesis of Stationarity against the Alternative of a Unit Root. *Journal of Econometrics*, no. 54(1–3), pp. 159–178.

[30] Nelson, C., Plosser, C. 1989. Trends and Random Walks in Macroeconomic Time Series. Some Evidence and Implications. *Journal of Monetary Economics*, vol. 10, pp. 139–162.

to real GDP series with and without a seasonal component are demonstrated in Table 1.

Table 1

Results of ADF and KPSS tests for the time series of real GDP of European countries

Country	WITH A SEASONAL COMPONENT						AFTER THE ELIMINATION OF THE SEASONAL COMPONENT						CF
	ADF		KPSS		trend		ADF		KPSS		trend		
	α	$\alpha+\beta t$	α	$\alpha+\beta t$	S	D	α	$\alpha+\beta t$	α	$\alpha+\beta t$	S	D	
Greece	0,671	0,442	I(1)**	I(1)***	2		0,954	0,413	I(1)**	I(1)**	2		α
Poland	0,691	0,912	I(1)***	I(1)*	2		0,818	0,835	I(1)***	I(1)**	2		α

The table shows p-values for the ADF test (H_0: unit root). For KPSS test (H_0: stationarity): I(1)*, I(1)** and I(1)***, denotes the rejection of the null hypothesis respectively at the significance level: 0,1; 0,05 and 0,01. Abbreviations: A-ADF; K-KPSS; 2-ADF and KPSS; S – stochastic; D – deterministic; α – drift; βt – linear trend.

Source: own calculations on the basis of Eurostat data.

The decision on whether to remove the drift (α) or the linear trend (βt) from the data before the application of the CF filter was based on the dominant number of tests that indicated the given option. The results are presented in CF column.

Finally, all three filters were used. The HP filter with variance penalising parameter λ equal to 1600 was applied to data without the seasonal component. The BK and CF filters were applied to both seasonally adjusted and not adjusted data. In both cases, the cyclical component of real GDP was defined as having a frequency between six and thirty two quarters (1.5 years to 8 years). For the BK filter, 12 observations were removed at the beginning and at the end of the analysed time series, as recommended by Baxter and King[31].

2.2. Comparison of results obtained with the use of HP, BK and CF filters

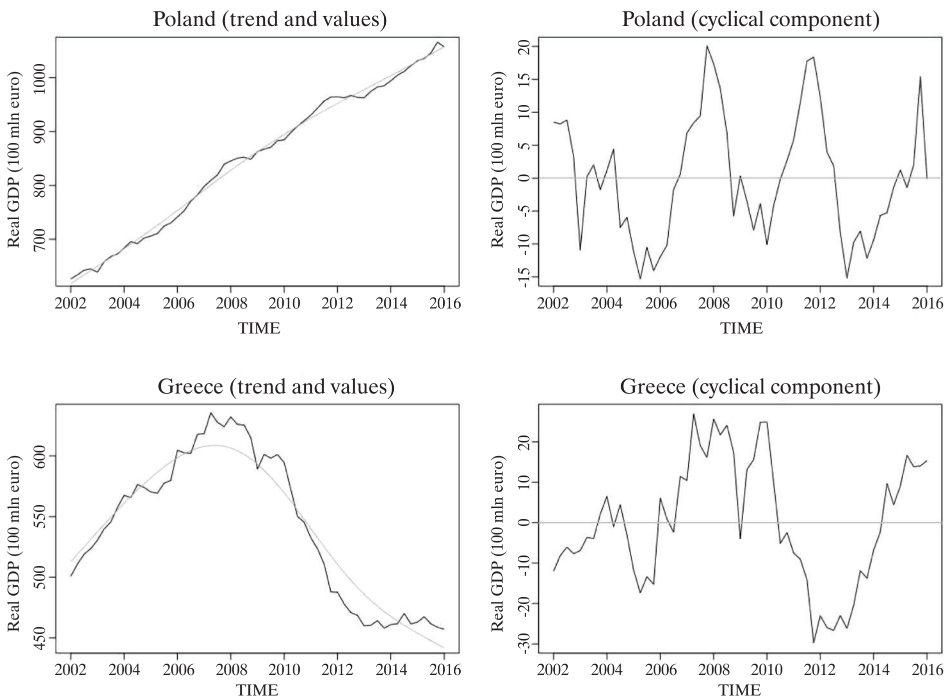
The results of applying the HP filter to the real GDP time series of Greece and Poland are shown in Figure 1. Panels a) and c) show real GDP values (black colour) and the trend (gray colour), while panels b) and d) represent

31 Baxter, M., King, R., *op. cit.*

the cyclical component. Values are expressed in hundreds of millions of euro from 2010.

Chart 1

Results of the application of the HP filter to the Greek and Polish to seasonally adjusted real GDP time series.



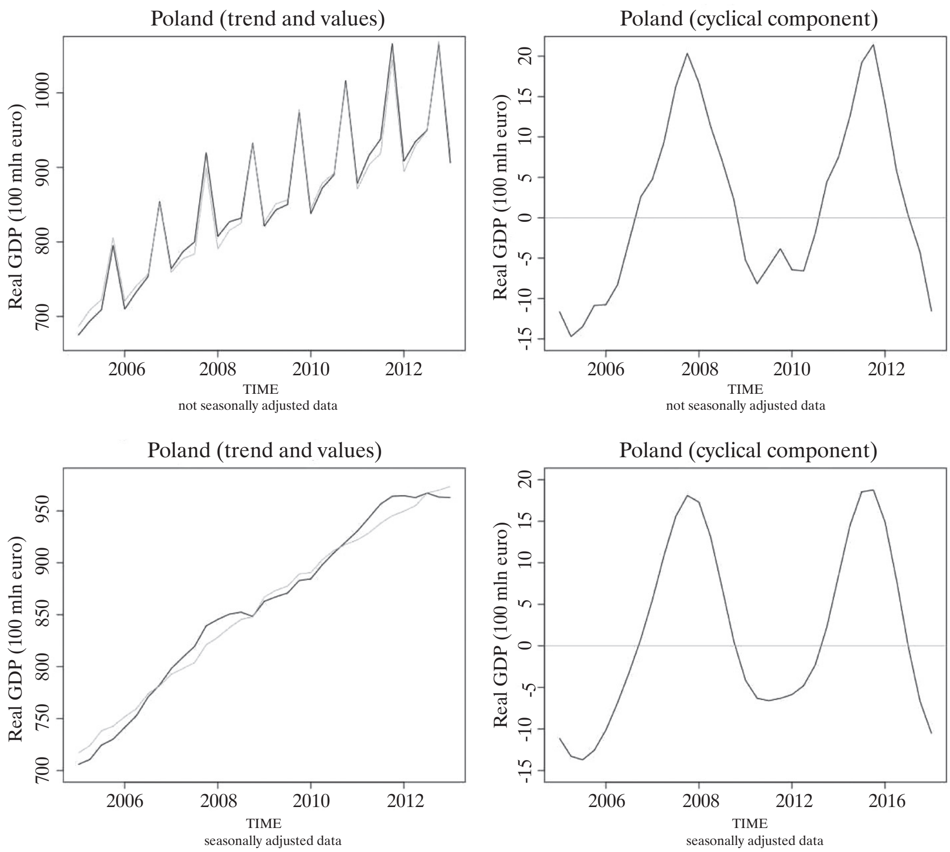
Source: own study on the basis of Eurostat data.

In Chart 1, it can be observed that in the case of Poland, the trend of GDP was characterised by an upward trend in the whole examined period. Poland was characterised by a relatively narrow band of cyclical fluctuations around the trend. The fall in GDP below the potential level from the second quarter of 2009 to the third quarter of 2010 and from the last quarter of 2012 to the end of 2014 is considered to be very low compared to the perturbations in the economies of other European countries. The case of Greece is significantly different from Poland. It can be seen that the decline in Greece's GDP did not result only from cyclical fluctuations, but was also caused by the decline in potential GDP. This can be explained by hysteresis, and no significant improvement in the Greek economy should be expected in the near future.

The above results were compared with results obtained on the basis of other filters. The BK filter was applied to the data with the seasonal component and to the seasonally adjusted data. The BK filter (like the CF) is a band pass filter. Thus, the result of using this filter is not a trend, but a non-cyclical component – the combination of a trend and a seasonal component. One way to obtain a better approximation of the trend by means of a non-cyclical component is the previous elimination of seasonal fluctuations. Charts 2 and 3 illustrate the effect of the application of the BK filter to the real GDP time series of Poland and Greece.

Chart 2

Results of the application of the BK filter to the real GDP time series of Poland

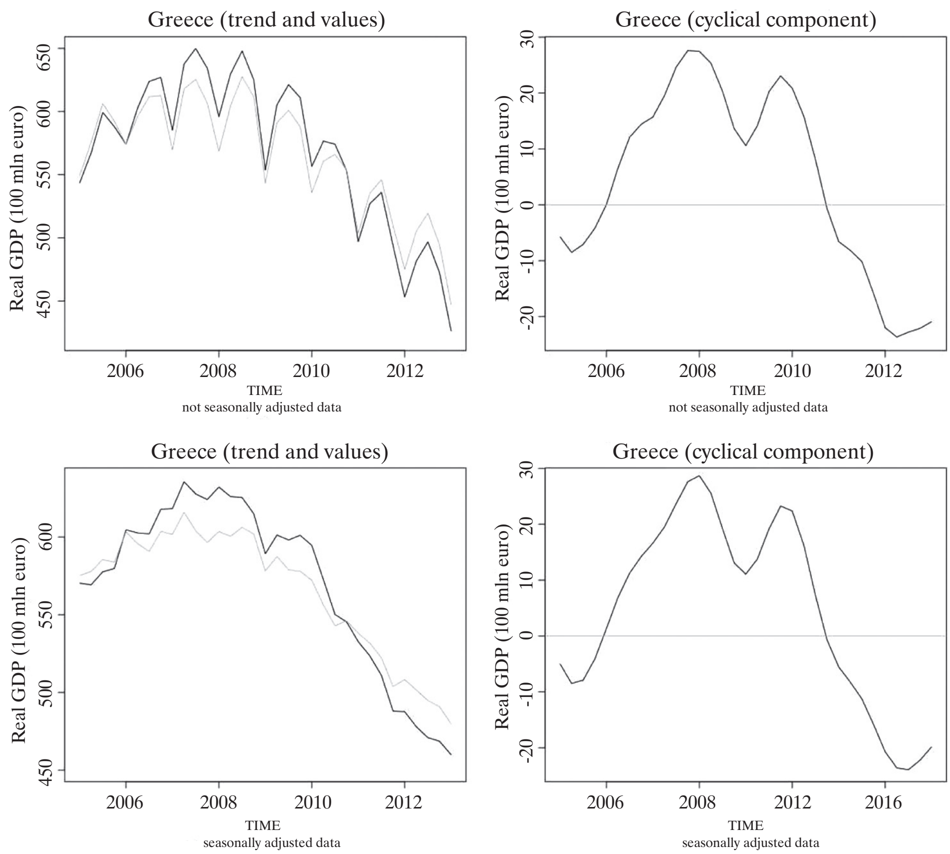


Source: own calculations on the basis of Eurostat data.

Comparing the results for the seasonally adjusted and not adjusted data for the trend leads to two main conclusions. First, the non-cyclical component is much smoother in the case of the data that do not include the seasonal component – thus they are much better approximations of the trend. Secondly, the cyclical component in both cases looks almost identical. This means that the BK filter copes very well with the elimination of seasonal fluctuations. The filter was programmed to retain only part of the series with a frequency of more than 1.5 years, that is, a frequency lower than seasonal fluctuations.

Chart 3

Results of the application of the BK filter to the real GDP time series of Greece

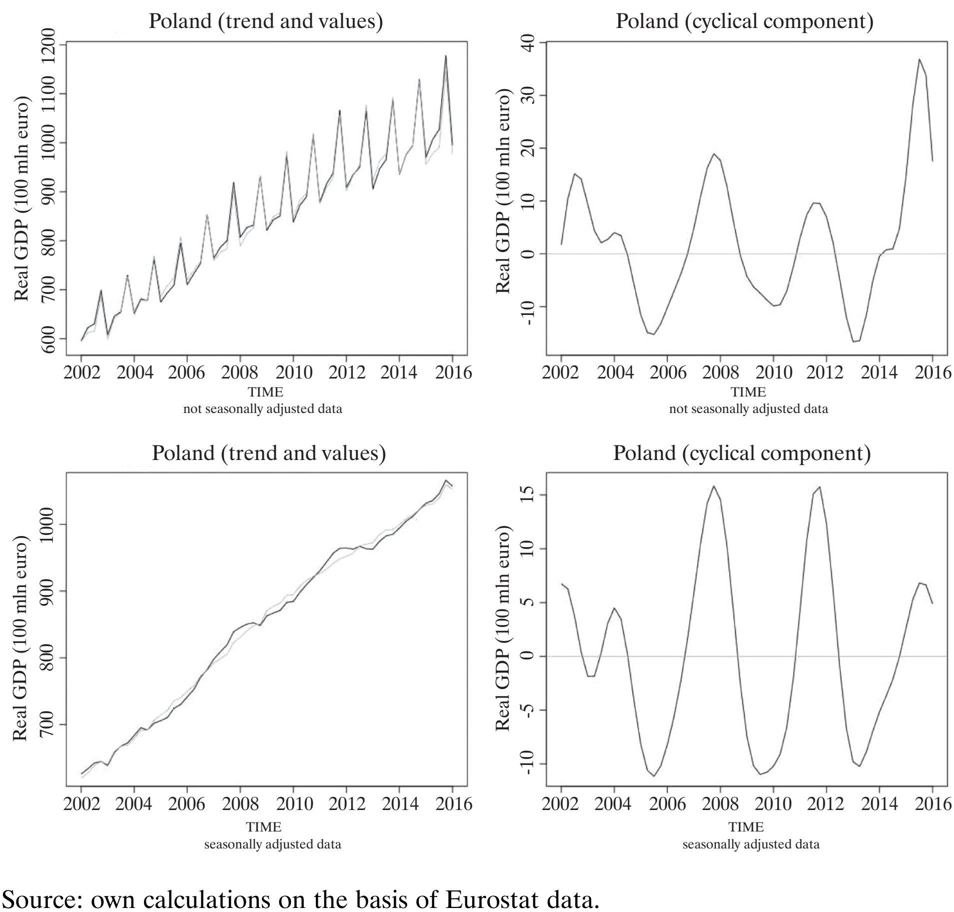


Source: own calculations on the basis of Eurostat data.

The application of the BK filter gave very similar results to those of the HP filter. The non-cyclical GDP component for Greece showed a declining trend, while Poland's GDP was characterised by a sustained increase of the non-cyclical component. On the other hand, the BK filter even for season-free data produces the non-cyclical component characterised by higher variability than in the case of the trend obtained as a result of the application of the HP. The opposite conclusion applies to the cyclical component that is much smoother than the results obtained with the BK filter. Unfortunately, the BK filter requires the removal of observations at the start and end of the analysed period. The biggest fluctuations are very similar to those obtained with the HP filter and are characterised by a phase shift not exceeding two quarters.

Chart 4

Results of the application of the CF filter to the time series of real GDP of Poland

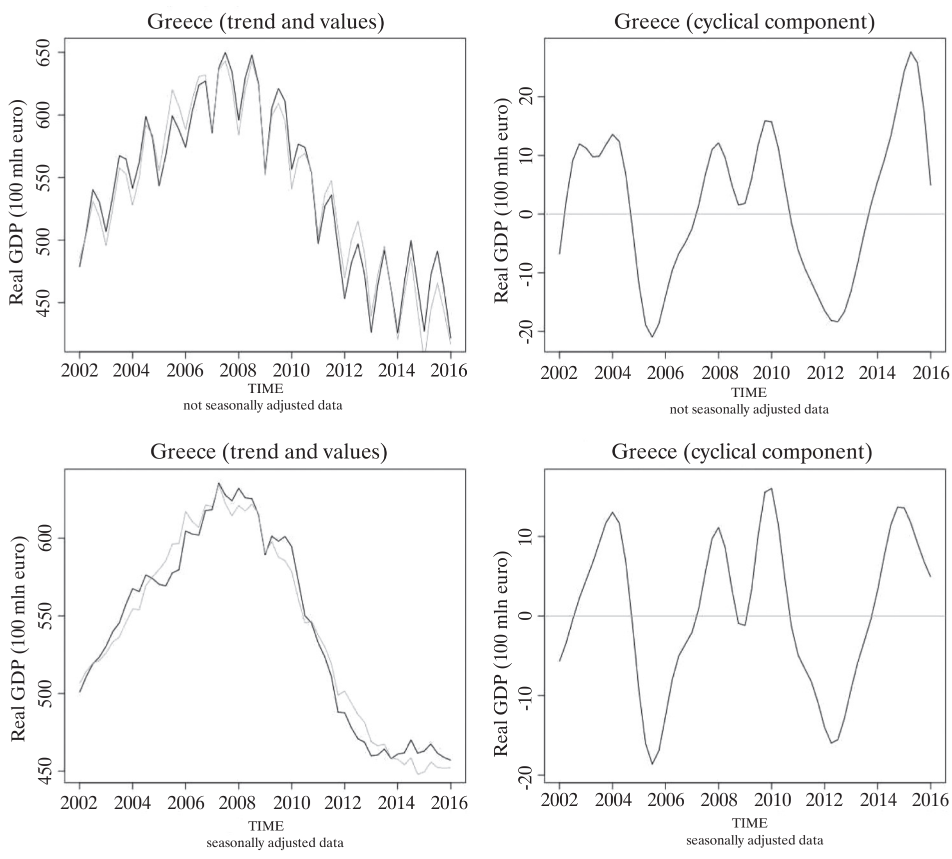


Source: own calculations on the basis of Eurostat data.

The results of the application of the CF filter to the time series of Poland and Greece are presented in Charts 4 and 5. The results in relation to the non-cyclical and cyclical component are the same as for the HP and the BK filters. The CF results for a non-cyclical component are characterised by higher variability than the HP trend, while cyclical fluctuations are smoother in the case of the CF filter. The phase shift again does not exceed two quarters. In the case of the CF filter the biggest differences are seen at the beginning and the end of the sample. This indicates that the use of the asymmetric moving average makes it possible to generate the full amount of observations, but the results at the beginning and the end of the sample are characterised by the greatest bias.

Chart 5

Results of the application of the CF filter to the time series of real GDP of Greece



Source: own calculations on the basis of Eurostat data.

The last possible comparison is the exemplification of the results obtained with the BK and the CF filters. The non-cyclical components obtained by means of both filters show no significant differences. This is not the case for cyclical components that are smoother for the CF filter in the case of the data containing a seasonal component. On the other hand, cyclical components for the season-free data look almost identical. The conclusion is that the CF filter copes better than the BK filter with data not adjusted for seasonality.

Conclusion

The first part of this paper presents the idea of spectral analysis and Hodrick-Prescott, Baxter-King and Christiano-Fitzgerald filters. The HP filter is a high pass filter, while the BK and the CF are band pass filters, and that is why they should be used in different situations. In addition, the BK filter imposes additional constraints as compared to the CF filter. The effect of these constraints is the lack of phase shifts and the stationarity of the obtained cyclical components, however at the expense of the inferior approximation of the ideal filter and the loss of observation at the end and beginning of the analysed time series.

Empirical applications showed very little difference in the operation of the filters. In particular, they indicate that the results obtained for the BK and the CF filters do not differ significantly from each other and are similar regardless of whether they are used for seasonally adjusted and not seasonally adjusted data. Although the BK and the CF filters should efficiently separate the cyclical component from the seasonal one, they work better when filtered data is seasonally adjuted. In the case of not adjusted data data, they mix the cyclical component with the seasonal one.

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Spectral analysis and the application of filters in the examination of business cycles

Summary

The spectral analysis enables division of a given time series into components characterised by a different frequency of fluctuations. Therefore, it is possible to use it for the extraction of the cyclical component from macroeconomic data. The most popular method of analysing components with a specific frequency is the use of Hodrick-Prescott, Baxter-King and Christiano-Fitzgerald filters, which are the subject of this paper. Empirical applications showed very little difference in the operation of the filters. In particular, they indicate that the results obtained for the BK and the CF filters do not differ significantly from each other and are similar regardless of whether they are used for data containing the seasonal component or not. Although the BK and the CF filters should efficiently separate the cyclical component from the seasonal one, they work better when filtered data is seasonally adjusted. In the case of not adjusted data, they mix the cyclical component with the seasonal one.

Analiza spektralna oraz zastosowanie filtrów w badaniu cykli koniunkturalnych

Streszczenie

Analiza spektralna pozwala podzielić dany szereg czasowy na komponenty charakteryzujące się różną częstotliwością wahań. Z tego względu możliwym jest jej wykorzystanie do ekstrakcji komponentu cyklicznego z danych makroekonomicznych. Najbardziej popularną metodą analizy komponentów o określonej częstotliwości jest zastosowanie filtrów Hodricka-Prescotta (HP), Baxter-Kinga (BK) oraz Christiano-Fitzgeralda (CF), które są przedmiotem tego artykułu. Zastosowania empiryczne wykazały bardzo niewielkie różnice w działaniu filtrów. W szczególności wskazują one, że wyniki uzyskane dla filtru BK oraz CF nie różnią się znacznie między sobą, oraz są podobne niezależnie od tego, czy zastosowano je dla danych zawierających komponent sezonowy czy danych oczyszczonych. Pomimo że filtry BK i CF powinny sprawnie oddzielać komponent cykliczny od sezonowego, działają one lepiej w sytuacji, gdy filtracji poddane są dane oczyszczone z komponentu sezonowego. W przypadku danych nieoczyszczonych, mieszają one komponent cykliczny z sezonowym.

Спектральный анализ и применение фильтров в исследовании бизнес-циклов

Резюме

Спектральный анализ позволяет произвести деление данных временных рядов на компоненты, для которых характерна различная частотность колебаний. По этой причине представляется возможным её использование с целью извлечения циклического компонента из макроэкономических данных. Наиболее популярным методом анализа компонентов с определённой частотностью является применение фильтров Ходрика-Прескотта (HP), Бакстера-Кинга (BK) и Кристиано-Фицджеральда (CF), которым посвящена данная статья. Эмпирические исследования их применения позволили выявить незначительные различия в действии фильтров. В частности, они показывают, что результаты, полученные для фильтра Бакстера-Кинга и Кристиано-Фицджеральда, различаются между собой в незначительной степени, и даже обладают сходством независимо от того, использованы ли они для данных с сезонным компонентом или очищенных данных. Несмотря на то, что фильтры Бакстера-Кинга и Кристиано-Фицджеральда должны чётко отделять циклический от сезонного компонента, они эффективнее действуют в ситуации, когда фильтрации подвержены данные, очищенные от сезонного компонента. В случае неочищенных данных, циклический компонент смешивается с сезонным.